Exact solution of Effective mass Schrödinger Equation for the Hulthen potential

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Abstract

A general form of the effective mass Schrödinger equation is solved exactly for Hulthen potential. Nikiforov-Uvarov method is used to obtain energy eigenvalues and the corresponding wave functions. A free parameter is used in the transformation of the wave function.

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1 Introduction

Quantum mechanical systems with position dependent effective mass (PDM) have been studied in different branches of physics by many authors [1,2,3,4,5,6,7]. Several authors have obtained the exact solutions of Schrödinger equation with position dependent mass [8-18]. Moreover, the Morse potential [19], one dimensional Coulomb-like potential [20], hard core potential [21], harmonic oscillator potential [22] are known as some real physical potentials that have been investigated within PDM framework.

In recent years many authors have been used Nikiforov-Uvarov (NU) approach for solving the Schrödinger equation (SE) [23,24,25,26,27,28,29].

In this work, the general form of PDEM Schrödinger equation is obtained by using a more general transformation of the wave function as $\varphi = m^{\eta}(x)\psi(x)$. NU approach is adapted to this general equation. Using an appropriate mass function, it is solved for Hulthen potential within this generalization. Energy eigenvalues and the corresponding wave functions are obtained. The contents of the paper is as follows: in section II, we introduce PDM approach and Nikiforov-Uvarov method. The next section involves solutions of the general PDM equation. Results are discussed in section IV.

2 Method

We write the one-dimensional effective mass Hamiltonian of the SE as [29]

$$H_{eff} = -\frac{d}{dx}\left(\frac{1}{m(x)}\frac{d}{dx}\right) + V_{eff}(x) \tag{1}$$

where V_{eff} has the form

$$V_{eff} = V(x) + \frac{1}{2}(\beta+1)\frac{m''}{m^2} - \left[\alpha(\alpha+\beta+1)+\beta+1\right]\frac{m'^2}{m^3} \tag{2}$$

with α, β are ambiguity parameters. Primes stand for the derivatives with respect to x and we have set $\hbar = 2m_0 = 1$. Thus the SE takes the form

$$\left(-\frac{1}{m}\frac{d^2}{dx^2} + \frac{m'}{m}\frac{d}{dx} + V_{eff} - E\right)\varphi(x) = 0 \tag{3}$$

We apply the following transformation

$$\varphi = m^{\eta}(x)\ \psi(x) \tag{4}$$

Hence, the SE takes the form

$$\left\{-\frac{d^2}{dx^2} - (2\eta - 1)\frac{m'}{m}\frac{d}{dx} - (\eta(\eta-2) + \alpha(\alpha+\beta+1) + \beta + 1)\frac{m'^2}{m^2} + (\frac{1}{2}(\beta+1) - \eta)\frac{m''}{m} + m(V - E)\right\}\psi = \tag{5}$$

Hulthen potential is given by [26]

$$V(x) = -V_0\frac{e^{-\lambda x}}{1 - qe^{-\lambda x}} \tag{6}$$

We give the following parameters including mass relation:

$$A^* = \alpha(\alpha+\beta+1) + \beta + 1 \tag{7}$$

$$m(x) = (1 - qe^{-\lambda x})^{-1} \tag{8}$$

$$\frac{m'}{m} = -q\lambda\frac{e^{-\lambda x}}{1 - qe^{-\lambda x}} \tag{9}$$

$$\frac{m''}{m} = q\lambda^2 e^{-\lambda x}\frac{1 + qe^{-\lambda x}}{(1 - qe^{-\lambda x})^2} \tag{10}$$

We introduce a variable changing in Eq.(5)given as

$$s = \frac{1}{1 - qe^{-\lambda x}} \tag{11}$$

and if we use Eqs.(6),(7),(8),(9),(10) and (11) in Eq.(5), it becomes,

$$\left(\frac{d^2}{ds^2} + \frac{2\eta - (2\eta+1)s}{s(1-s)}\frac{d}{ds} + \frac{1}{s^2(1-s)^2}(-\xi_1 s^2 + \xi_2 s - \xi_3)\right)\psi = 0. \tag{12}$$

Parameters defined in Eq.(12) have the following form:

$$
\begin{aligned}
-\xi_1 &= (\eta(\eta-2)+A^*)-2(\frac{1}{2}(\beta+1)-\eta)+\frac{V_0}{q\lambda^2} && (13)\\
\xi_2 &= -2(\eta(\eta-2)+A^*)+3(\frac{1}{2}(\beta+1)-\eta)-\frac{V_0}{q\lambda^2}+\frac{E}{\lambda^2} && (14)\\
-\xi_3 &= \eta(\eta-2)+A^*-(\frac{1}{2}(\beta+1)-\eta). && (15)
\end{aligned}
$$

where $V(s)=\frac{V_0}{q}(1-s)$. Now, we apply the NU method starting from its standard form

$$\psi_n^{''}(s)+\frac{\tilde{\tau}(s)}{\sigma(s)}\psi^{'}(s)+\frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi_n(s)=0. \tag{16}$$

Comparing Eqs.(12) and (16), we obtain

$$\sigma=s,\tilde{\tau}(s)=3-4\eta,\tilde{\sigma}(s)=-\xi_1 s^2-\xi_2 s+\xi_3 \tag{17}$$

In the NU method, the function π and the parameter λ are defined as [23]

$$\pi(s)=\frac{\sigma^{'}-\tau(s)}{2}\pm\sqrt{\left(\frac{\sigma^{'}-\tau(s)}{2}\right)^2-\tilde{\sigma}(s)+k\sigma(s)} \tag{18}$$

and

$$\lambda=k+\pi^{'} \tag{19}$$

To find a physical solution, the expression in the square root must be square of a polynomial. Then, a new eigenvalue equation for the SE becomes

$$\lambda=\lambda_n=-n\tau^{'}-\frac{n(n-1)}{2}\sigma^{''}(s),(n=0,1,2,...) \tag{20}$$

where

$$\tau(s)=\tilde{\tau}(s)+2\pi(s) \tag{21}$$

and it should have a negative derivative [23]. A family of particular solutions for a given λ has hypergeometric type of degree. Thus, $\lambda=0$ will corresponds to energy eigenvalue of the ground state, i.e. $n=0$. The wave function is obtained as a multiple of two independent parts:

$$\psi(s)=\phi(s)y(s) \tag{22}$$

where $y(s)$ is the hypergeometric type function written with a weight function ρ as

$$y_n(s) = \frac{B_n}{\rho(s)} \frac{d^n}{ds} [\sigma^n(s)\rho(s)] \tag{23}$$

where $\rho(s)$ must satisfy the condition [23]

$$(\sigma\rho)^{'} = \tau\rho \tag{24}$$

The other part is defined as a logarithmic derivative

$$\frac{\phi^{'}(s)}{\phi(s)} = \frac{\pi(s)}{\sigma(s)} \tag{25}$$

3 Solutions

If we take Eq.(12) into account, comparing with Eq.(16), it is observed that $\tilde{\tau} = 2\eta - (2\eta+1)s$, $\sigma = s(1-s)$, $\tilde{\sigma} = -\xi_1 s^2 + \xi_2 s - \xi_3$. Using $z = \frac{1}{2}(1-2\eta)$, one obtains

$$\pi = z(1-s) \pm \begin{cases} (\sqrt{\xi_1 - \xi_2 + \xi_3} - \sqrt{\xi_3 + z^2})s + \sqrt{\xi_3 + z^2}, & k_1 = \xi_2 - 2\xi_3 + 2\zeta; \\ (\sqrt{\xi_1 - \xi_2 + \xi_3} + \sqrt{\xi_3 + z^2})s - \sqrt{\xi_3 + z^2}, & k_2 = \xi_2 - 2\xi_3 - 2\zeta. \end{cases} \tag{26}$$

where $\zeta = \sqrt{\xi_3(\xi_1 - \xi_2 + \xi_3 + z^2) - z^2(\xi_2 - \xi_1)}$. Now we can introduce $\tau(s)$ as given below,

$$\tau(s) = \begin{cases} 1 - 2s + 2((\sqrt{\xi_1 - \xi_2 + \xi_3} - \sqrt{\xi_3 + z^2})s + \sqrt{\xi_3 + z^2}) \\ 1 - 2s - 2((\sqrt{\xi_1 - \xi_2 + \xi_3} - \sqrt{\xi_3 + z^2})s + \sqrt{\xi_3 + z^2}) \\ 1 - 2s + 2((\sqrt{\xi_1 - \xi_2 + \xi_3} + \sqrt{\xi_3 + z^2})s - \sqrt{\xi_3 + z^2}) \\ 1 - 2s - 2((\sqrt{\xi_1 - \xi_2 + \xi_3} + \sqrt{\xi_3 + z^2})s - \sqrt{\xi_3 + z^2}) \end{cases} \tag{27}$$

Derivative of $\tau(s)$ is obtained as

$$\tau^{'} = \begin{cases} -2 + 2(\sqrt{\xi_1 - \xi_2 + \xi_3} - \sqrt{\xi_3 + z^2}) \\ -2 - 2(\sqrt{\xi_1 - \xi_2 + \xi_3} - \sqrt{\xi_3 + z^2}) \\ -2 + 2(\sqrt{\xi_1 - \xi_2 + \xi_3} + \sqrt{\xi_3 + z^2}) \\ -2 - 2(\sqrt{\xi_1 - \xi_2 + \xi_3} + \sqrt{\xi_3 + z^2}) \end{cases} \tag{28}$$

Here, first derivative of τ should be $\tau^{'} < 0$ in order to obtain physical solutions. Thus we choose k and our functions which help us to derive the energy eigenvalues and eigenfunctions:

$$k = \xi_2 - 2\xi_3 - 2\sqrt{\xi_3(\xi_1 - \xi_2 + \xi_3 + z^2) - z^2(\xi_2 - \xi_1)} \tag{29}$$

$$\tau = 1 - 2s - 2[(\sqrt{\xi_1 - \xi_2 + \xi_3} + \sqrt{\xi_3 + z^2})s - \sqrt{\xi_3 + z^2}] \tag{30}$$

$$\pi = z(1-s) - [(\sqrt{\xi_1 - \xi_2 + \xi_3} + \sqrt{\xi_3 + z^2})s - \sqrt{\xi_3 + z^2}] \tag{31}$$

$$\tau^{'} = -2 - 2(\sqrt{\xi_1 - \xi_2 + \xi_3} + \sqrt{\xi_3 + z^2}). \tag{32}$$

Using Eq.(19), the relation given below

$$\lambda = z^2 - z + \xi_1 - (\sqrt{\xi_1 - \xi_2 + \xi_3} + \sqrt{\xi_3 + z^2})^2 - (\sqrt{\xi_1 - \xi_2 + \xi_3} + \sqrt{\xi_3 + z^2}) \tag{33}$$

is obtained. With the aid of Eq.(20), this equality can be written:

$$\lambda = \lambda_n = -n(-2 - 2(\sqrt{\xi_1 - \xi_2 + \xi_3} + \sqrt{\xi_3 + z^2})) + n(n-1) \tag{34}$$

Substituting $\Lambda = \sqrt{\xi_1 - \xi_2 + \xi_3} + \sqrt{\xi_3 + z^2}$, Λ can be written

$$\Lambda = \frac{1}{2}\left(-(2n+1) \pm \sqrt{1+4\gamma}\right) \tag{35}$$

where $\gamma = \xi_1 + z(z-1)$. Now let us discuss two cases here depending on signs of Λ.

Case 1:

$$\sqrt{\xi_1 - \xi_2 + \xi_3} + \sqrt{\xi_3 + z^2} = \frac{1}{2}\left(-(2n+1) + \sqrt{1+4\gamma}\right) \tag{36}$$

then, ξ_3 is obtained as

$$\xi_3 = \left(\frac{\xi_2 - \xi_1 + z^2}{2n+1-\sqrt{1+4\gamma}} + \frac{1}{4}\left(2n+1-\sqrt{1+4\gamma}\right)\right)^2 \tag{37}$$

Using the definitions of ξ_1, ξ_2 and ξ_3, E_n is given by

$$E_n = -\frac{\lambda^2}{4}\left(2n+1-\sqrt{1+4\gamma} - 2\sqrt{-\eta(\eta-1) - A^* + \frac{\beta+1}{2}}\right)^2 - \lambda^2(\eta - \frac{1}{2})^2 \tag{38}$$

Case 2:

$$\sqrt{\xi_1 - \xi_2 + \xi_3} + \sqrt{\xi_3 + z^2} = \frac{1}{2}\left(-(2n+1) - \sqrt{1+4\gamma}\right) \tag{39}$$

then, ξ_3 reads

$$\xi_3 = \left(\frac{\xi_2 - \xi_1 + z^2}{2n+1+\sqrt{1+4\gamma}} - \frac{1}{4}\left(2n+1+\sqrt{1+4\gamma}\right)\right)^2 \tag{40}$$

Energy eigenvalues can be written:

$$E_n = \frac{\lambda^2}{4}\left(2n+1+\sqrt{1+4\gamma} + 2\sqrt{-\eta(\eta-1) - A^* + \frac{\beta+1}{2}}\right)^2 - \lambda^2(\eta - \frac{1}{2})^2 \tag{41}$$

Using Eqs(24) and (25), ϕ and ρ are obtained as

$$\phi = s^{z+\sqrt{\xi_3+z^2}}(1-s)^{\sqrt{\xi_1-\xi_2+\xi_3}} \tag{42}$$

and

$$\rho(s) = s^{2\sqrt{\xi_3+z^2}}(1-s)^{2\sqrt{\xi_1-\xi_2+\xi_3}} \tag{43}$$

Solution of y can be obtained from Eq.(23):

$$y_n(s) = P_n^{(2\sqrt{\xi_3+z^2},2\sqrt{\xi_1-\xi_2+\xi_3})}(1-2s). \tag{44}$$

Hence, the wave function has the following form:

$$\psi_n = s^{z+\sqrt{\xi_3+z^2}}(1-s)^{\sqrt{\xi_1-\xi_2+\xi_3}}P_n^{(2\sqrt{\xi_3+z^2},2\sqrt{\xi_1-\xi_2+\xi_3})}(1-2s) \tag{45}$$

If $z+\sqrt{\xi_3+z^2}<0$ and $\sqrt{\xi_1-\xi_2+\xi_3}>0$, it is required that $|z+\sqrt{\xi_3+z^2}| \geq \sqrt{\xi_1-\xi_2+\xi_3}$ and if $\sqrt{\xi_1-\xi_2+\xi_3}<0$, $z+\sqrt{\xi_3+z^2}>0$, $|\sqrt{\xi_1-\xi_2+\xi_3}| \geq z+\sqrt{\xi_3+z^2}$ for physical solutions.

4 Conclusions

NU method adapted solutions are obtained for Hulthen potential within PDEM Schrödinger equation. We have proposed a transformation of the wavefunction in a general form that leads to solutions of well-known eigenvalues and eigenfunctions of Hulthen potential. Furthermore, energy relations of the mass independent equation are obtained for two cases.

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